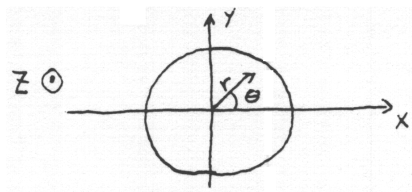
University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

**SOLUTION TO PROBLEM SET 10**

*Solutions by P. Pebler*

**1** *Purcell 6.26* *A round wire of radius $r_o$ carries a current $I$ distributed uniformly over the cross section of the wire. Let the axis of the wire be the $z$ axis, with $\hat{\mathbf{z}}$ the direction of the current. Show that a vector potential of the form $\mathbf{A} = A_o(x^2 + y^2)\hat{\mathbf{z}}$ will correctly give the magnetic field $\mathbf{B}$ of this current at all points inside the wire. What is the value of the constant $A_o$?*




The magnetic field is the curl of the vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A} = 2A_o y\,\hat{\mathbf{x}} - 2A_o x\,\hat{\mathbf{y}}$$

If we use plane polar coordinates in the $x - y$ plane,

$$\mathbf{B} = 2A_o r(\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}}) = -2A_o r\,\hat{\phi} \quad .$$

We know that the magnetic field circles in the counterclockwise direction for a current coming out of the page. We can find the magnitude from Ampere's law.

$$2\pi r B = \frac{4\pi}{c} I \frac{r^2}{r_o^2}$$

$$B = \frac{2I}{cr_o^2} r$$

The vector potential $\mathbf{A}$ therefore works with the constant

$$A_o = -\frac{I}{cr_o^2} \quad .$$

**2** *Purcell 6.28* *A proton with kinetic energy $10^{16}$ eV ($\gamma = 10^7$) is moving perpendicular to the interstellar magnetic field which in that region of the galaxy has a strength $3 \times 10^{-6}$ gauss. What is the radius of curvature of its path and how long does it take to complete one revolution?*

Magnetic forces do no work. They can only change the direction of the momentum. Because the force is perpendicular to the velocity, we can instantaneously think about the motion as being along a circle of some radius $R$. Because the field and velocity are perpendicular, the magnitude of the force is

$$F = \frac{evB}{c} \quad .$$

If this were a non-relativistic problem, we could find the radius $R$ by equating the force with $mv^2/R$. In the relativistic case, this formula turns out to be correct with the replacement $m \to \gamma m$, but this is something that must be proved. If we wait a time $\Delta t$, the momentum will swing through some angle $\Delta\theta$, and $|\Delta\mathbf{p}| = p\Delta\theta$. (Please note that $\Delta p$ is not the same thing as $|\Delta\mathbf{p}|$.) This angle $\Delta\theta$ will also be the angle of the circle we go through in this time. Therefore, in the infinitesimal limit $\Delta \to d$, $v = \omega r$. Consequently,

$$\left|\frac{d\mathbf{p}}{dt}\right| = p\frac{d\theta}{dt} = p\omega = \frac{pv}{R} \quad .$$

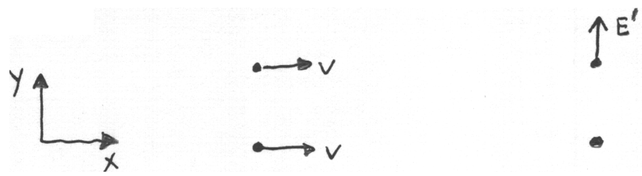
Equating this with the force,

$$R = \frac{pc}{eB} \simeq \frac{\gamma mc^2}{eB} = \frac{10^7(1.5\times 10^{-3}\ ergs)}{(4.8\times 10^{-10}\ esu)(3\times 10^{-6}\ gauss)} = 1\times 10^{19}\ cm \quad .$$

The period is

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi R}{v} \simeq \frac{2\pi R}{c} = 2.1\times 10^9\ s \quad .$$

**3** *Purcell 6.32 Two electrons move along parallel paths, side by side, with the same speed $v$. The paths are a distance $r$ apart. Find the force acting on one of them in two ways. First, find the force in the rest frame of the electrons and tranform this force back to the lab frame. Second, calculate the force from the fields in the lab frame. What can be said about the force between them in the limit $v \to c$?*




In the particle rest frame, the field is just the Coulomb field and the force magnitude is $e^2/r^2$. If we use the transformation formulas (14) in Purcell, the primed frame must be the particle rest frame.

$$\mathbf{F} = \frac{1}{\gamma}\frac{e^2}{r^2}\,\hat{\mathbf{y}}$$

To find the fields in the lab frame, it is easiest to transform them back from the rest frame where

$$\mathbf{E}' = -\frac{e}{r^2}\,\hat{\mathbf{y}} \qquad\qquad \mathbf{B}' = 0 \quad .$$

Please note that most transformation formulas found in books assume that the primed frame is moving in the positive $x$ direction of the unprimed frame. If you wish to use these formulas verbatim, you must choose your frames correctly. Here the particles are going to the right so the rest frame is the primed frame. To switch back to the lab frame, we need the inverse of the equations (6.60) in Purcell. We can accomplish this by simply switching the primes and the sign of $\boldsymbol{\beta}$. Then

$$\mathbf{E}_{\|} = \mathbf{E}'_{\|} = \mathbf{0} \quad ,$$

$$\mathbf{E}_{\perp} = \gamma(\mathbf{E}'_{\perp} - \boldsymbol{\beta}\times\mathbf{B}'_{\perp}) = \gamma\mathbf{E}'_{\perp} = -\gamma\frac{e}{r^2}\,\hat{\mathbf{y}} \quad ,$$

$$\mathbf{B}_{\|} = \mathbf{B}'_{\|} = \mathbf{0} \quad ,$$

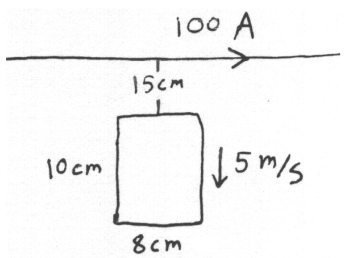
$$\mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \boldsymbol{\beta} \times \mathbf{E}'_\perp) = -\gamma\beta\frac{e}{r^2}\,\hat{\mathbf{z}} \quad .$$

The force is then

$$\mathbf{F} = (-e)\mathbf{E} + \frac{(-e)}{c}\mathbf{v} \times \mathbf{B} = \gamma\frac{e^2}{r^2}\,\hat{\mathbf{y}} - \gamma\beta^2\frac{e^2}{r^2}\,\hat{\mathbf{y}} = \gamma\frac{e^2}{r^2}(1-\beta^2)\,\hat{\mathbf{y}} = \frac{1}{\gamma}\frac{e^2}{r^2}\,\hat{\mathbf{y}} \quad .$$

In the limit $v \to c$, we see that $\mathbf{F} \to \mathbf{0}$.

**4** *Purcell 7.4* *Calculate the electromotive force in the moving loop in the figure at the instant when it is in the position there shown. Assume the resistance of the loop is so great that the effect of the current in the loop itself is negligible. Estimate very roughly how large a resistance would be safe, in this respect. Indicate the direction in which current would flow in the loop, at the instant shown.*




We first calculate the flux. We will define the positive direction to be into the page. The field is that of a wire. The current is given in SI, so we must use SI formulas.

$$\int \mathbf{B} \cdot d\mathbf{a} = \int_x^{x+L} \frac{\mu_o I}{2\pi r} w\,dr = \frac{\mu_o I w}{2\pi} \ln\frac{x+L}{x}$$

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{\mu_o I w}{2\pi}\frac{x}{x+L}\left(\frac{v}{x} - \frac{(x+L)v}{x^2}\right) = \frac{\mu_o I w}{2\pi}\frac{Lv}{x(x+L)}$$

$$\mathcal{E} = 2.1 \times 10^{-5}\ V$$

By choosing into the page as positive for flux, we have also defined clockwise as the positive way to go around the loop. Since $\mathcal{E}$ is positive, the induced current will be clockwise.

**5** *Purcell 7.9* *Derive an approximate formula for the mutual inductance of two circular rings of the same radius $a$, arranged like wheels on the same axle with their centers a distance $b$ apart. Use an approximation good for $b \gg a$.*

From Purcell Eq. 6.41 (where $a$ and $b$ are interchanged relative to this problem) the field along the axis of a ring is

$$B_z = \frac{2\pi a^2 I}{c(a^2 + z^2)^{3/2}} \quad .$$

We may use the information $b \gg a$, and approximate the $z$ component of the field everywhere in the second loop as

$$B_z = \frac{2\pi a^2 I}{c(a^2 + b^2)^{3/2}} \simeq \frac{2\pi a^2 I}{cb^3} \quad ,$$

so the flux is

$$\Phi_B = \frac{2\pi a^2 I}{cb^3}\pi a^2 = \frac{2\pi^2 a^4}{cb^3} I \quad .$$

The induced emf is then

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi_B = -\frac{2\pi^2 a^4}{c^2 b^3}\frac{dI}{dt} \quad ,$$

and the mutual induction is

$$M = \frac{2\pi^2 a^4}{c^2 b^3} \quad ,$$

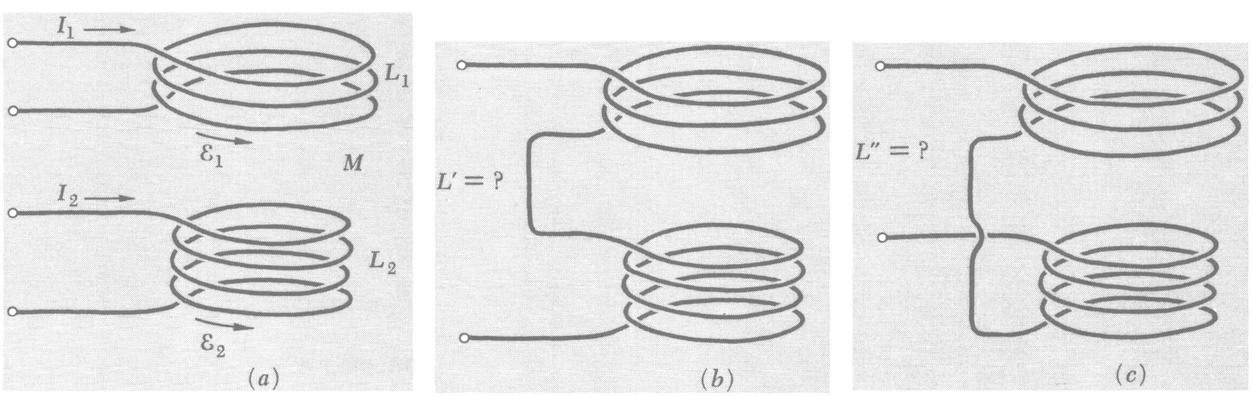
in cgs units. If you use SI formulas this becomes

$$M = \frac{\mu_o}{4\pi}\frac{2\pi^2 a^4}{b^3} \quad .$$

**6** *Purcell 7.11 Two coils with self-inductances $L_1$ and $L_2$ and mutual inductance $M$ are shown with the positive direction for current and electromotive force indicated. The equations relating currents and emf's are*

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} \pm M\frac{dI_2}{dt} \qquad\qquad \mathcal{E}_2 = -L_2\frac{dI_2}{dt} \pm M\frac{dI_1}{dt} \quad .$$

*Given that $M$ is always to be taken as positive, how must the signs be chosen in these equations? What if we had chosen the other direction for positive current and emf in the lower coil? Now connect the two coils together as in b. What is the inductance $L'$ of this circuit? What is the inductance $L''$ of the circuit formed as shown in c? Which circuit has the greater self-inductance? Considering that the self-inductance of any circuit must be a positive quantity, see if you can deduce anything concerning the relative magnitudes of $L_1$, $L_2$, and $M$.*




Imagine first that the current $I_2$ is positive and increasing so that $dI_2/dt > 0$. In this case the magnetic field due to coil 2 will point up through coil 1. As the current $I_2$ increases, the field it creates will increase and the flux up through coil 1 will increase. By using Lenz's law, we find we need an induced current that will create a magnetic field that will oppose this *change* in the flux. In this case, the field should point down through coil 1. To do this the induced current must flow in the negative direction as it is defined for coil 1. Thus, the induced emf must be negative and we need the negative sign. The same argument will tell you to choose the negative sign in the second equation also. (You should go through it yourself however.)

If the sign convention for coil 2 had been switched, the same argument would switch the sign in both equations. (Do it yourself though.)

With the circuit in $b$, since both emf positive directions point in the same way, the total emf across the new circuit is

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} - L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \quad .$$

We also have $I = I_1 = I_2$ so that

$$\mathcal{E} = -(L_1 + L_2 + 2M)\frac{dI}{dt} \quad ,$$

and the self inductance is

$$L' = L_1 + L_2 + 2M \quad .$$

With the circuit in $c$, the sign conventions "conflict" so that

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} \quad ,$$

but with $I = I_1 = -I_2$ so that

$$\mathcal{E} = -(L_1 + L_2 - 2M)\frac{dI}{dt} \quad ,$$

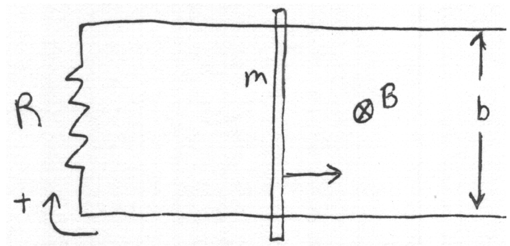
and the self inductance is

$$L'' = L_1 + L_2 - 2M \quad .$$

If the self inductance of a coil were negative, the circuit would be unstable – any change in current would result in more current the same direction which would built indefinitely. Therefore we must have $L'' > 0$ and

$$M \leq \frac{L_1 + L_2}{2} \quad .$$

**7** *Purcell 7.14  A metal crossbar of mass $m$ slides without friction on two long parallel conducting rails a distance $b$ apart. A resistor $R$ is connected across the rails at one end; compared with $R$, the resistance of bar and rails is negligible. There is a uniform field* $\mathbf{B}$ *perpendicular to the plane of the figure. At time $t = 0$ the crossbar is given a velocity $v_o$ toward the right. What happens then? Does the rod ever stop moving? If so, when? How far does it go? How about conservation of energy?*




Let us assume the magnetic field is into the page, and let's make that positive so that clockwise is positive for the loop. The flux is then

$$\Phi_B = bxB \quad .$$

The emf (in SI) is

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -bvB = IR \quad ,$$

so the current is counterclockwise. The bar will feel a force due to the magnetic charges moving through it. The force is

$$\mathbf{F} = |I|\mathbf{L} \times \mathbf{B} = -|I|bB\,\hat{\mathbf{x}} \quad .$$

We can solve for the motion using $F = ma$.

$$m\frac{dv}{dt} = -|I|bB = -\frac{b^2B^2}{R}v$$

$$v(t) = v_o e^{-b^2B^2t/mR}$$

The bar never stops moving. It will approach the distance

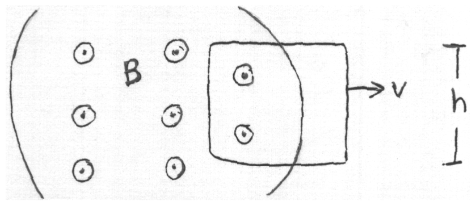
$$d = \int_0^\infty v_o e^{-b^2B^2t/mR}dt = \frac{mRv_o}{b^2B^2} \quad .$$

The lost kinetic energy is dissipated by the resistor.

$$I = \frac{b}{B}Rv$$

$$U = \int_o^\infty I^2Rdt = \int_o^\infty \frac{b^2B^2}{R}v_o^2e^{-2b^2B^2t/mR} = \frac{b^2B^2}{R}v_o^2\frac{mR}{2b^2B^2} = \frac{1}{2}mv_o^2$$

**8** *Purcell 7.16 The shaded region represents the pole of an electromagnet where there is a strong magnetic field perpendicular to the plane of the paper. The rectangular frame is made of* 5 *mm diameter aluminum. suppose that a steady force of* 1 *N can pull the frame out in* 1 *s. If the force is doubled, how long does it take? If the frame is made of* 5 *mm brass, with about twice the resistivity, what force is needed to pull it out in* 1 *s? If the frame were* 1 *cm diameter aluminum, what force is needed to pull it out in* 1 *s? Neglect inertia of the frame and assume it moves with constant velocity.*




If we assume a constant velocity, the force necessary to pull out the loop will be equal in magnitude to the magnetic force on the loop. We will ignore signs here. The net force will be on the left wire of the frame.

$$F = |I|hB$$

The magnetic flux will be something like $\Phi = LxB$, where $x$ is the length of loop in the field. Then the emf is

$$|\mathcal{E}| = \left|-\frac{d}{dt}\Phi\right| = hvB \quad .$$

The current is $|I| = |\mathcal{E}|/R$ and the resistance is $R = \rho L/A = \rho L/\pi r^2$, where $L$ is the total length of the loop and $r$ is the radius of the wire of which it is made.

$$F = hBhBv\frac{A}{\rho L} \propto \frac{vr^2}{\rho}$$

If the force is doubled, the speed doubles and it takes half the time or 0.5 $s$. If the resistivity doubles with the same speed, the force is halved so that $F = 1$ $N$. If the radius doubles with the same speed, the force is four times as great or 4 $N$.